CHAPTER V

AVERAGE FORCES.

Let us first use equation (44), the zk component of the force acting along the direction of the axis of the first atom at the origin, to obtain its average value when the atom's axes are turned in all possible ways. Select any representative atom and locate it permanently at the origin O (Fig. 2) with axis along the k coordinate axis. Imagine the center of the second atom at P with its axis along the k' coordinate axis, making an angle alpha with the k-axis.

For a first integration revolve the k'-axis of the second atom around the line Pk, keeping the angle alpha constant, the motion being like that of a conical pendulum. Then integrate for one complete revolution.

But, because of the way the coordinate axes have been defined, making the j-axis always perpendicular to the plane $k'Pk$ (the direction of the skew product $k' \times k$), both the x- and y-axes revolve with the atom. The line PE equal to z (or to Z when $r = 1$) remains constant during the revolution, but the point G moves around a circle in the equatorial plane of the first atom with diameter OE. The cosine of the angle POG between the x-axis and the radius vector, r, is X by definition. This is the only variable quantity in either (44) or (23) for the first integration, the conical movement of the k' axis. In Fig. 2 we see that

$$OE = r \cos POE. \tag{45}$$

But

$$\cos OPE = z/r = Z. \tag{46}$$

Hence $\quad \cos POE = (1 - Z^2)^{1/2} \quad$ and $\quad OE = r(1 - Z^2)^{1/2}$ (47)

Also $\quad x = OE \cos GOE = r(1 - Z^2)^{1/2} \cos GOE$ (48)

or $\quad X = (1 - Z^2)^{1/2} \cos GOE.$ (49)

The average value of $\cos GOE$ for one revolution is zero, hence the average value of X is zero.

By the use of Table V, page 43, in the book, the definite integrals of the powers of a cosine for a complete revolution may be found. All the odd powers of the cosine average to zero, but not the even powers.

Average $\quad X^2 = \frac{1}{2}(1 - Z^2)$ (50)

Average $\quad X^4 = \frac{3}{8}(1 - Z^2)^2 \quad$ (See (87), (88), (89) in the book) (51)

Beginning now with equation (44) for the first average, the r^{-3} term within the brace becomes

$$(3MZ - m)r^{-3} = [3(gX + mZ)Z - m]r^{-3} = (3Z^2 - 1)mr^{-3},$$

which agrees with (99) in the book. The r^{-4} term of (44) for the first term gives

$$\begin{aligned} -\tfrac{15}{2}M^2Z &= -\tfrac{15}{2}(g^2X^2 + m^2Z^2 + 2gmXZ)Z \\ &= -\tfrac{15}{2}g^2(\tfrac{1}{2})(Z - Z^3) - \tfrac{15}{2}m^2Z^3 \\ -\tfrac{15}{2}M^2Z &= -\tfrac{15}{4}g^2Z + \tfrac{15}{2}(\tfrac{1}{2}g^2 - m^2)Z^3. \\ +\tfrac{3}{2}NZ &= \tfrac{3}{2}(g^2 + m^2)Z \\ +3Mm &= +3(gX + mZ)m = 3m^2Z. \end{aligned}$$

The sum of the three terms gives for the Z terms

$$\tfrac{9}{4}(2m^2 - g^2)Zr^{-4},$$

and for the Z^3 terms,

$$-\tfrac{15}{4}(2m^2 - g^2)Z^3r^{-4},$$

thus reducing (44), including the r^{-4} term, to

$$\begin{aligned} F_{zk} = \frac{E_1E_2}{k}\{&-Zr^{-2} + (3Z^2 - 1)mr^{-3} \\ &+ [\tfrac{9}{4}(2m^2 - g^2)Z - \tfrac{15}{4}(2m^2 - g^2)Z^3]r^{-4} \cdots \}k, \end{aligned} \tag{52}$$

which is equation (99) in the book.

This equation expresses the force upon a stationary charge E_1 in