

value of τ_2 selected, namely 3 terms in this example. So, for any fixed value of τ_2 we have

$$R_\infty = h \sum_{\tau=1}^{\tau=\infty} \nu = hK \sum_1^{\tau_2} \frac{1}{\tau^2}, \quad (13)$$

giving τ all values from 1 to τ_2 .

Substituting (13) in (9) gives

$$V_0 - h \sum_{\tau=1}^{\tau=\infty} \nu = V_\infty, \quad \text{the same constant for every value of } \tau_2. \quad (14)$$

Therefore we must seek for some expression for V_0 that will reduce the difference in (14) to the same constant for all values of τ_2 . The complementary quantity μ satisfies this requirement, where, see (4),

$$\mu = K \left[\frac{1}{\tau^2} + \frac{1}{(\tau + \tau_2)^2} \right]. \quad (15)$$

When $\tau_2 = 3$, for example,

$$\begin{aligned} & h \sum_{\tau=1}^{\tau=\infty} \mu \\ &= hK \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots \right). \end{aligned} \quad (16)$$

Adding (16) and (12), the last series in each cancels, giving according to (14)

$$\begin{aligned} V_0 - h \sum_{\tau=1}^{\tau=\infty} \nu &= V_\infty = -h \left(\sum_{\tau=1}^{\tau=\infty} \mu + \sum_{\tau=1}^{\tau=\infty} \nu \right) \\ &= -2hK \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right). \end{aligned} \quad (18)$$

But this infinite series is known, and sums to exactly $\pi^2/6$. Hence we have found the value of the constant difference between the first terms of (14) or of (18), and the value of V_∞ , namely

$$V_\infty = -hK\pi^2/3, \quad \text{a constant for all values of } \tau_2. \quad (19)$$

Hence by (14) and (19) we find

$$V_0 = V_\infty + h \sum_{\tau=1}^{\tau=\infty} \nu = -hK \left(\frac{\pi^2}{3} - \sum_1^{\tau_2} \frac{1}{\tau^2} \right). \quad (20)$$

For specific values of τ_2 (equation (4)) we obtain

$$\tau_2 = 1; \quad V_0 = -hK \left(\frac{\pi^2}{3} - \frac{1}{1^2} \right) = -.4932 \times 10^{-10} \text{ erg.} \quad (21)$$

$$\tau_2 = 2; \quad V_0 = -hK \left[\frac{\pi^2}{3} - \left(1 + \frac{1}{2^2} \right) \right] = -.4394 \times 10^{-10} \text{ erg.} \quad (22)$$

$$\begin{aligned} \dots \dots \dots \\ \tau_2 = \infty; \quad V_0 &= -hK \left(\frac{\pi^2}{3} - \frac{\pi^2}{6} \right) = -hK\pi^2/6 \\ &= -.3543 \times 10^{-10} \text{ erg.} \end{aligned} \quad (23)$$

The energy per electron is one half of these values. It is customary, however, to express such energies in terms of the voltage required to give an electron these kinetic energies. To convert into practical units, volts, multiply one half the above energies* by $\frac{c}{e} 10^{-8} = .6284 \times 10^{12}$ approximately, and thus obtain

the ionizing voltages given in Table I at the beginning of Chapter XVIII. They agree with experimental observations.

The amount of energy radiated in one excursion according to (13) is

τ_2	R_∞	
1	1	$\times hK$
2	1.25	"
3	1.361	"
4	1.423	"
.....
∞	1.644,934 = $\pi^2/6$	"

The maximum value corresponds to the curve of highest frequencies in Fig. 1 (the last curve) showing frequencies dying away from K down to zero.

* One volt equals $10^8/c$ electrostatic units, where c is the velocity of light.

CHAPTER XIX

PROOF OF A NEW PROPOSITION IN SOLID GEOMETRY.

If through any point four lines be drawn making equal angles, each with any other, and if from a second point at a fixed distance from the first point four perpendiculars are drawn, one to each of the said four lines, then the sum of the squares of these four perpendiculars is constant for all points at the same distance from the first point.

A corollary is that the sum of the squares of the projections of the radius vector to the second point upon the four said lines is also constant. A projection of the radius vector is the distance from the first point to the foot of the said perpendicular along one of the four lines.

* * *

It should first be proved that the system of four lines making equal angles, each with any other, must be lines parallel to the four diagonals of a cube, or the four medial lines of a regular tetrahedron. The cosine of the angle between any two of the four lines is $\pm\frac{1}{3}$, and the angle is approximately $70^\circ 32'$ or $109^\circ 28'$, according to whether the acute or obtuse angle is intended. (A proof of this need not be given here.)

The proposition to be proved may then be stated in other words. If from a point at a fixed distance from the center of a cube four perpendiculars are drawn, one to each diagonal of the cube, then the sum of the squares of these four perpendiculars remains constant, irrespective of the orientation of the cube.

Referring to Figure 26, let the diagonals of the cube be $A'A$, $B'B$, $C'C$ and $D'D$, its edges being $A'B$, $A'C$ and $A'D$. Assume a set of rectangular unit vectors, i , j and k , and make the cube edge unity. Then

$$A'C = i; \quad A'D = j \quad \text{and} \quad A'B = k.$$

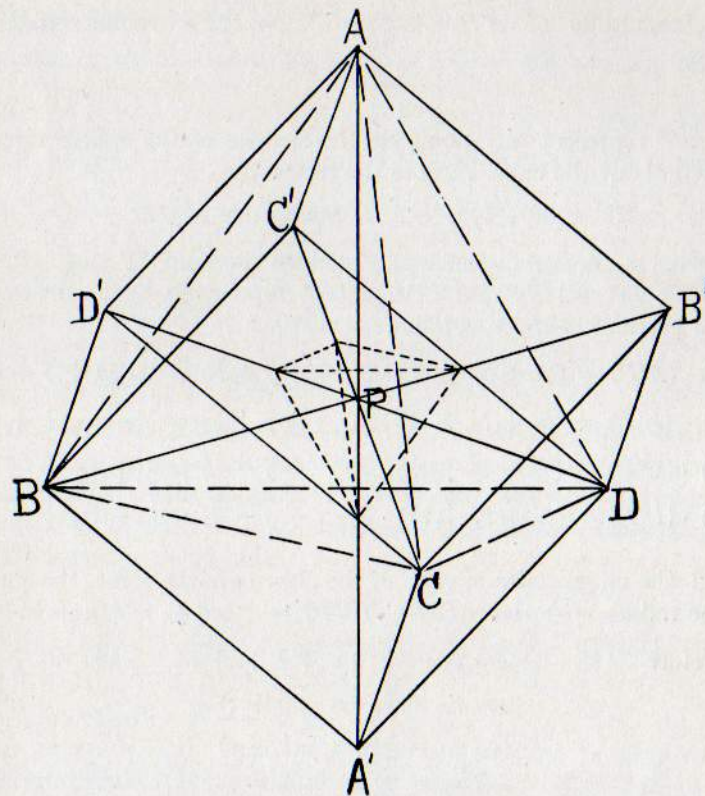


FIG. 26.

$$\begin{aligned} \text{Hence } A'A &= +i + j + k & \text{and } A'P &= \frac{1}{2}(+i + j + k) \\ B'B &= -i - j + k & \text{“ } B'P &= \frac{1}{2}(-i - j + k) \\ C'C &= +i - j - k & \text{“ } C'P &= \frac{1}{2}(+i - j - k) \\ D'D &= -i + j - k & \text{“ } D'P &= \frac{1}{2}(-i + j - k) \end{aligned}$$

$$(A'A) \cdot (A'A) = (A'A)^2 = 1 + 1 + 1 = 3; \quad (A'P)^2 = \frac{1}{4}(A'A)^2 = \frac{3}{4};$$

$$A'P = \frac{1}{2}\sqrt{3}.$$

